3.3 Cramer's Rule, Volume and Linear Transformations

Cramer's Rule assists particularly for:

1) theoretical calculations

2) geometric interpretation of determinants

Cramer's Rule

Important for study of sensitivity of $\mathbf{x}$ to changes in $\mathbf{b}$

Definition:

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \ \cdots \ \underset{\substack{\uparrow \\ \text{col } i}}{\mathbf{b}} \ \cdots \ \mathbf{a}_n]$$

Theorem, Cramer's Rule: Let $A_{n\times n}$ be invertible, then unique solution of $A\mathbf{x} = \mathbf{b}$ is:

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n$$

Proof:

$$\begin{aligned} A \cdot I_i(\mathbf{x}) &= A[\mathbf{e}_1 \ \cdots \ \mathbf{x} \ \cdots \ \mathbf{e}_n] \\ &= [A\mathbf{e}_1 \ \cdots \ A\mathbf{x} \ \cdots \ A\mathbf{e}_n] \end{aligned}$$

$$= [\mathbf{a}_1 \ \cdots \ \mathbf{b} \ \cdots \ \mathbf{a}_n]$$
$$= A_i(\mathbf{b})$$

By multiplicative property of determinants:

$$(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$$

Using cofactor expansion of $I_i(x)$ along row i, its determinant equals x_i . . . divide by non-zero $\det A$ . . . ■

Example: Use Cramer's rule to solve:

$$3\,x_1 - 2\,x_2 = 6$$
$$-5\,x_1 + 4\,x_2 = 8$$

Solution:

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix},$$

Since $\det A = 2$, system has unique $\mathbf{x}$, and:

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{24 + 16}{2} = 20$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{24 + 30}{2} = 27$$

Application to Engineering

A Laplace transform (from electrical engineering and control theory) converts approximate system of linear differen-

tial equations to system of linear algebraic equations, with a parameter in the coefficients

Example: Consider the following system with unspecified parameter s:

$$\begin{aligned} 3s\,x_1 - 2\,x_2 &= 4 \\ -6\,x_1 + s\,x_2 &= 1 \end{aligned}$$

Determine values of s for a unique solution and describe solution using Cramer's rule.

Solution:

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix},$$

and:

$$\det A = 3s^2 - 12 = 3(s+2)(s-2)$$

$\therefore \exists!\mathbf{x}$ when $s \neq \pm 2$, and:

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{4s+2}{3(s+2)(s-2)}$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}$$

Determinants as Area or Volume

Theorem: $|\det A|$ in

1) $\boldsymbol{R}^2$ is area of parallelogram

2) $\boldsymbol{R}^3$ is volume of parallelepiped

of columns of A

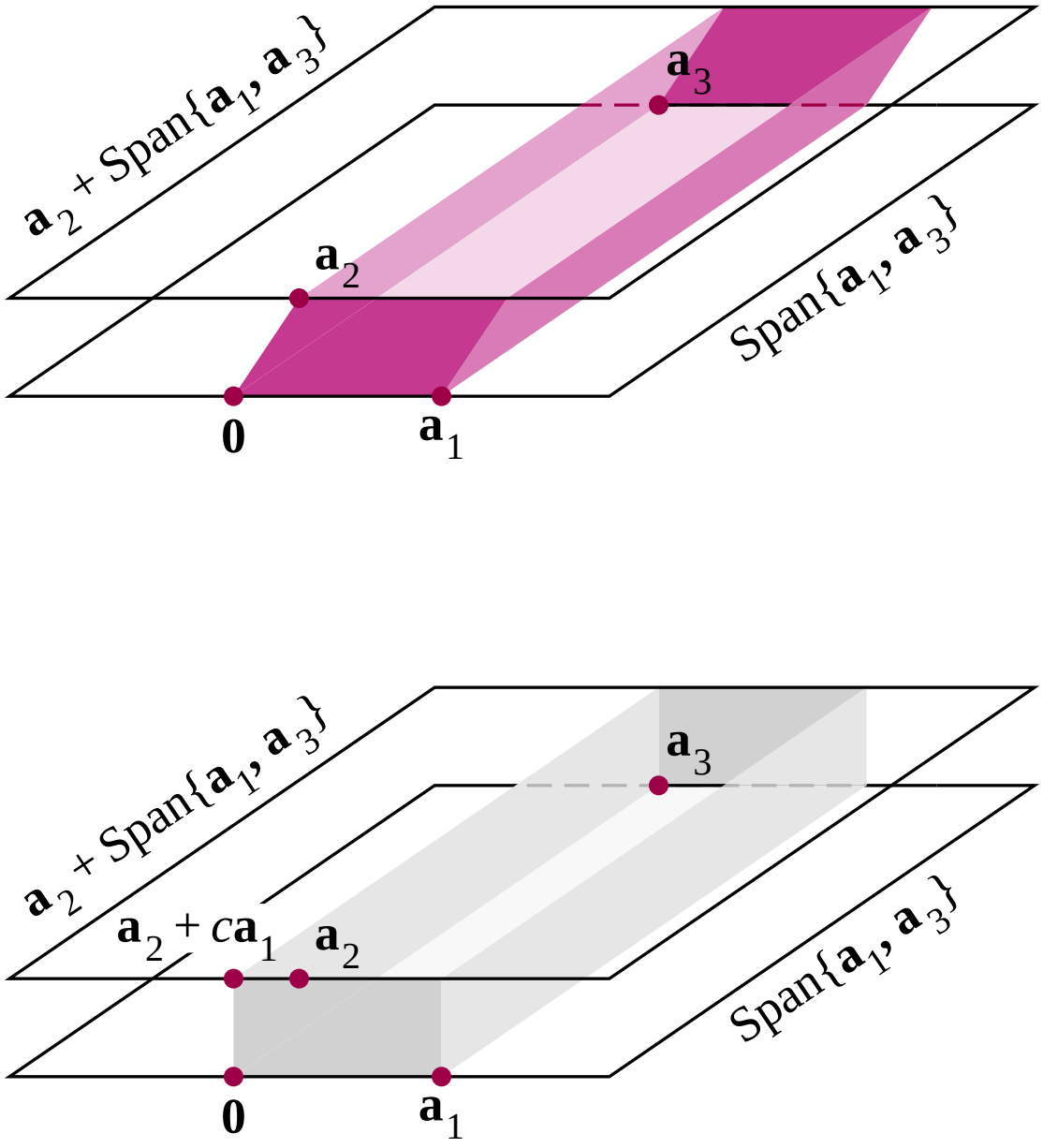
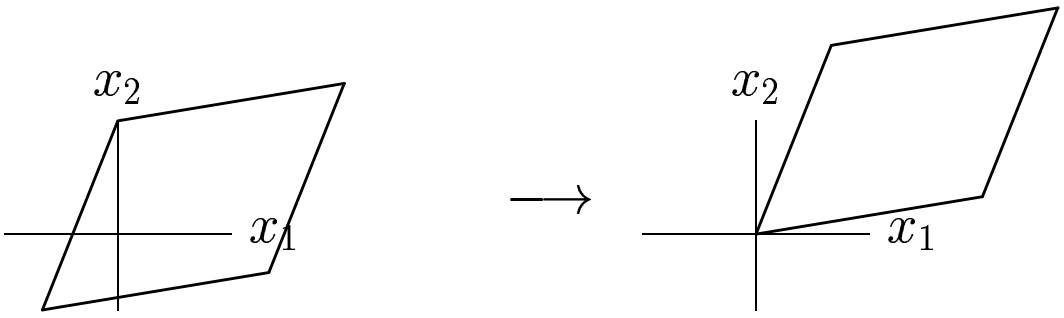


FIGURE 4 Two parallelepipeds of equal volume.

Lay, *Linear Algebra and Its Applications*, Second Edition—Update
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Example: Calculate area of parallelogram bounded by $(-2, -2)$, $(0, 3)$, $(4, -1)$, $(6, 4)$.

Solution: Translate parallelogram to have origin as vertex by subtracting one vertex from all, e.g., $(-2, -2)$. Now: parallelogram bounded by $(0, 0)$, $(2, 5)$, $(6, 1)$, $(8, 6)$.



The area is now determined by $|\det A| = 28$, as: $A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$

(Actually, $\begin{bmatrix} 2 & 8 \\ 5 & 6 \end{bmatrix}$ and $\begin{bmatrix} 8 & 6 \\ 6 & 1 \end{bmatrix}$ also work. Why?)

Linear Transformations

Theorem: Let T be the linear transformation with standard matrix $A_{2\times 2}$, and S a parallelogram in $\boldsymbol{R}^2$, then:

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

If $A_{3\times 3}$, and S a parallelepiped in $\boldsymbol{R}^3$, then:

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

Theorem: Generalizing, the same holds for regions of finite area in $\boldsymbol{R}^2$ and finite volume in $\boldsymbol{R}^3$.

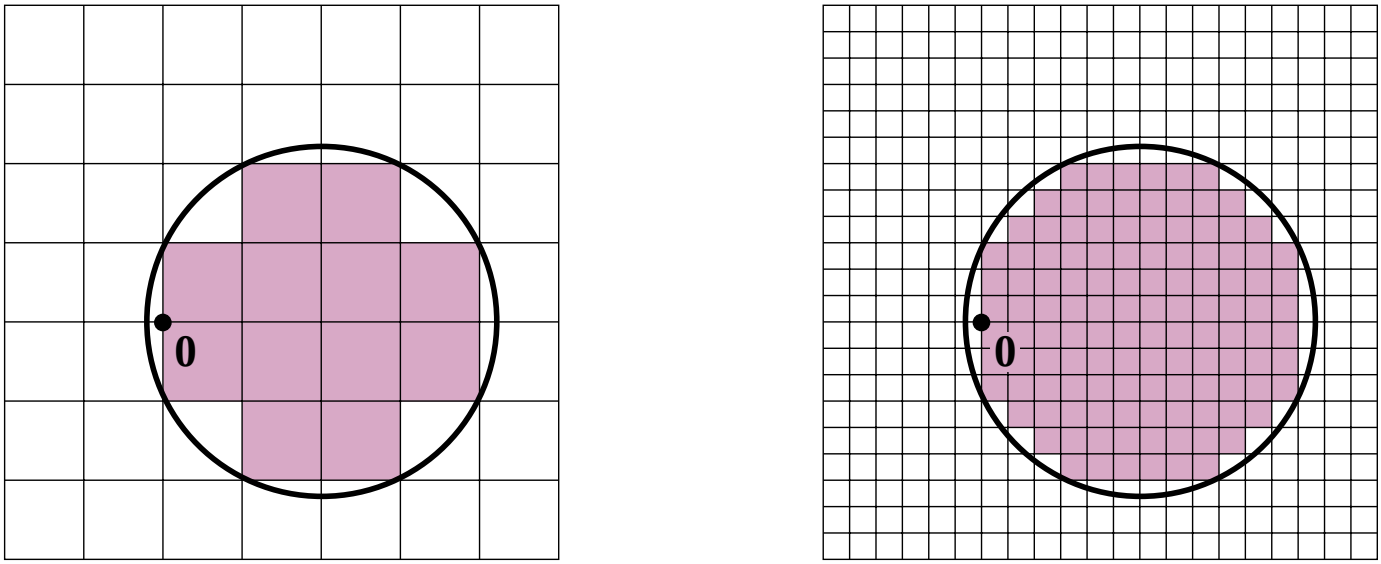


FIGURE 6 Approximating a planar region by squares. The approximation improves as the grid becomes finer.

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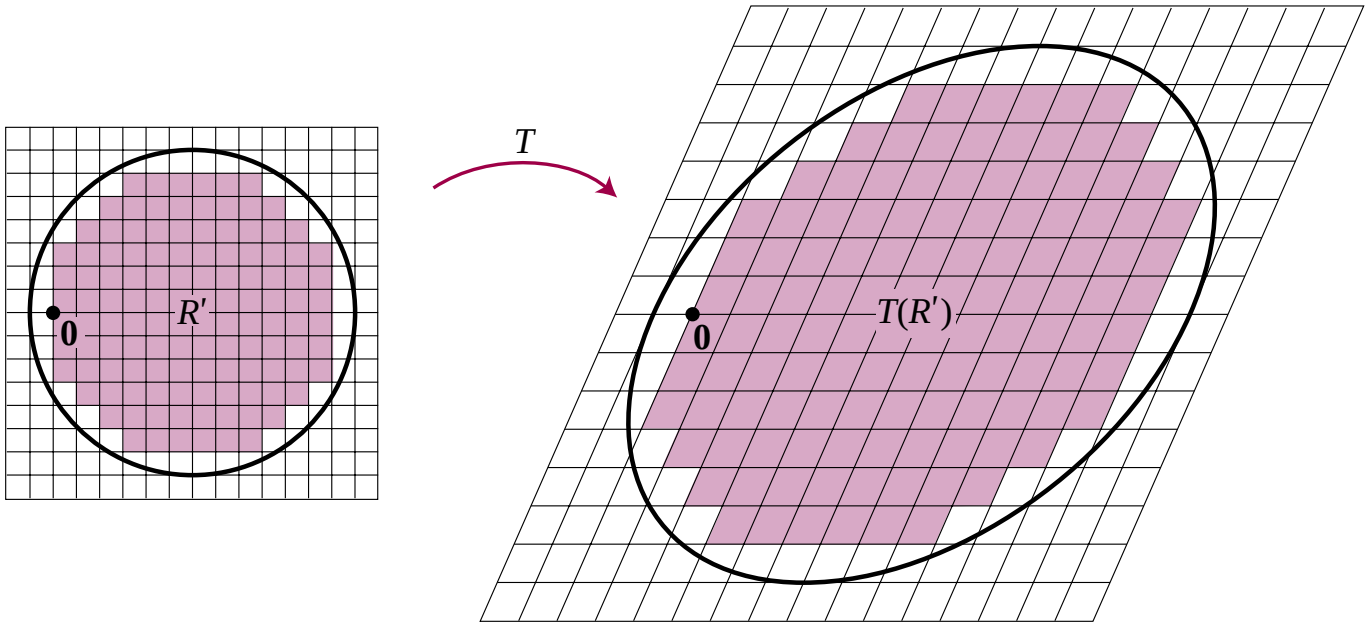


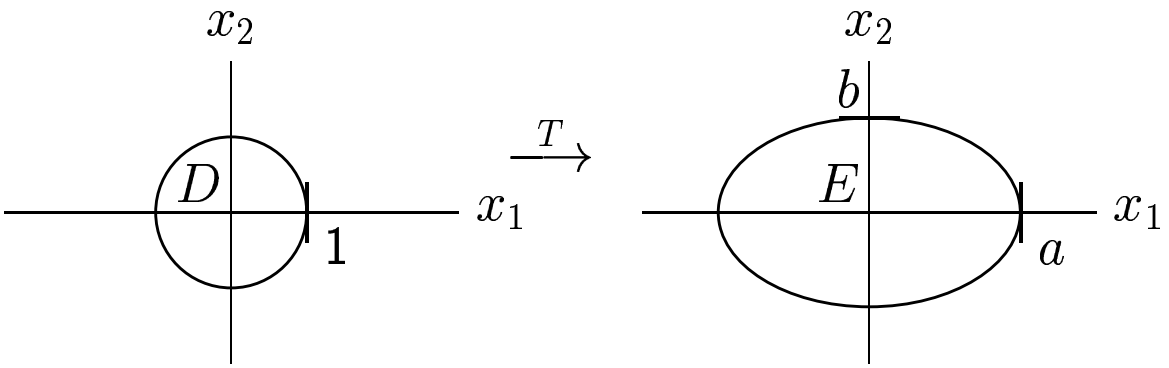
FIGURE 7 Approximating $T(R)$ by a union of parallelograms.

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Example: Let a and b be positive numbers. Find the area of of the region E bounded by the ellipse whose equation is:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

Solution: We claim that E is the image of the unit disk D under T with matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (Consider $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.)



Therefore:

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi(1)^2 = \pi ab \end{aligned}$$