2.5 Matrix Factorizations

Matrix factorization: write A as product of 2 or more matrices

The LU Factorization

Often have, e.g., in time-dependent problems:

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad \ldots, \quad A\mathbf{x} = \mathbf{b}_p$$

One option: compute (if exists) A^{-1}, then solutions are:

$$A^{-1}\mathbf{b}_1, \quad A^{-1}\mathbf{b}_2, \quad \ldots, \quad A^{-1}\mathbf{b}_p$$

But: LU factorization: $^{1)}$less work, $^{2)}$more accurate

Assume $A_{m\times n}$ (requires no row interchanges and) can be written as LU with:

1) $L_{m\times m}$ unit lower triangular

2) $U_{m\times n}$ echelon form of A

so, e.g.:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So instead of solving $A\mathbf{x} = \mathbf{b}$, define $\mathbf{y} := U\mathbf{x}$ and solve:

$$\boxed{\begin{array}{rcl} L\mathbf{y} & = & \mathbf{b} \\ U\mathbf{x} & = & \mathbf{y} \end{array}}$$

Isn't this more work?? No! Because L and U are triangular.

E.g.:

$$\begin{aligned}
[L \;\; \mathbf{b}] &= \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \;\; \mathbf{y}]
\end{aligned}$$

Then, the backward substitution:

$$
\begin{aligned}
[U \;\; \mathbf{y}] &= \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 3 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = [I \;\; \mathbf{x}]
\end{aligned}
$$

An LU Factorization Algorithm

Assumption (can be removed later): only adding multiples of

rows to lower ones

Row echelon form obtained by p replacements:

$$E_p \cdots E_1 A = U$$

Then

$$A = (E_p \cdots E_1)^{-1} U \equiv LU$$

Products and inverses of unit lower triangular matrices remain unit lower triangular

Note: $E_p \cdots E_1$ transform A to U and L to I (why?)

Lower entries of L are the multipliers of E_i, i.e., multipliers of pivot rows, to zero out columns below pivots

E.g., LU factorization of:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Since A is 4×5, L is 4×4:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Reduction to row echelon form:

$$\begin{aligned} A &= \begin{bmatrix} \boxed{2} & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & \boxed{3} & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{bmatrix} = U \end{aligned}$$

Therefore:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Verify: $A = LU$!

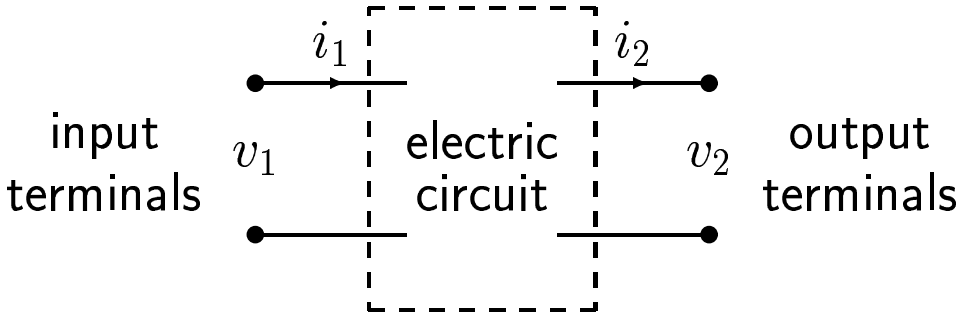
Numerical notes: for dense $A_{n \times n}$ with $n \geq 30$:

1) LU factorization: $2n^3/3$ flops; calculation A^{-1}: $2n^3$ flops
2) solving with L and U: $2n^2$ flops
3) multiplication of $A^{-1}\mathbf{b}$: $2n^2$ flops; *however* A^{-1} calculation and $A^{-1}\mathbf{b}$ multiplication *both* less exact
4) if A is sparse $\Rightarrow$ L and U likely too, but not A^{-1}

A Matrix Factorization in Electrical Engineering

Problem: constructing an electrical network with specified properties

Note: a *different* (i.e., non-LU) type of factorization

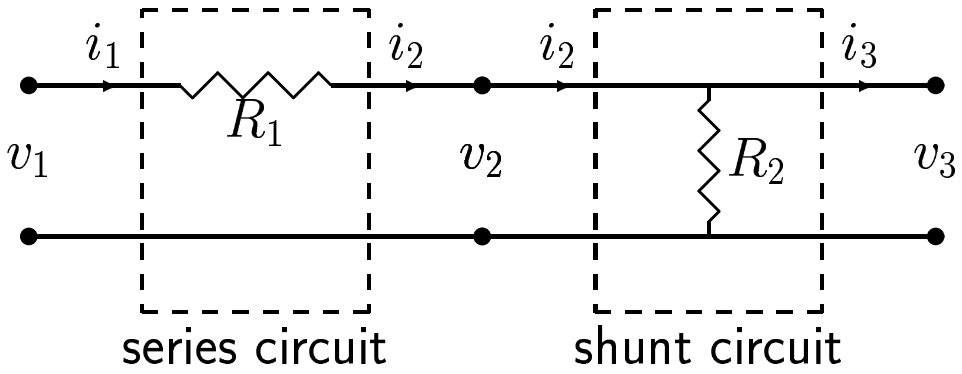


where input and output, resp., voltage (volts) and current (amps): $\begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$, $\begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$

Often transformation is linear, with *transfer matrix* A:

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} \mapsto \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}, \text{ is: } \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$$

Ladder network ($\geq$ 2 circuits in series):



R_1 and R_2 are resistance (in ohms)

Transfer matrices:

series circuit and shunt circuit

$$\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}$$

Problem:

1) calculate transfer matrix for ladder network above

2) design new ladder network with transfer matrix: $\begin{bmatrix} 1 & -8 \\ -.5 & 5 \end{bmatrix}$

Solution:

1) note order:

$$A_2 A_1 = \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix}$$

2) want R_1 and R_2 such that:

$$\begin{bmatrix} 1 & -R_1 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -.5 & 5 \end{bmatrix}$$

therefore: $R_1 := 8$ ohms and $R_2 := 2$ ohms