1.6 Applications of Linear Systems

Many everyday problems are set up as linear systems.

Possible solution sets:

1) no solutions
2) one unique solution
3) an infinite number of solutions

Applications:

1) a homogeneous system in economics
2) balancing chemical equations
3) network flow

A Homogeneous System in Economics

Leontief's "exchange model":

1) economy sectors: coal, electric, steel
2) known: output from each sector to each sector
3) unknown: total output prices for each sector

4) "equilibrium prices": as output prices, such that incomes balance expenses

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

Unknowns: find (if possible) p_C, p_E, $p_S \ni$ income match expenditures

Solution:

Columns represent sector's output (totals to 100%)

Rows represent sector's input (expenditures)

Setting up equations for equilibrium (i.e., balance).

For Coal:

$$p_C = .4p_E + .6p_S$$

For Electricity:

$$p_E = .6p_C + .1p_E + .2p_S$$

For Steel:

$$p_S = .4p_C + .5p_E + .2p_S$$

Rearranging we arrive at *homogeneous* system:

$$\begin{aligned} p_C - .4\,p_E - .6\,p_S &= 0 \\ -.6\,p_C + .9\,p_E - .2\,p_S &= 0 \\ -.4\,p_C - .5\,p_E + .8\,p_S &= 0 \end{aligned}$$

Row reduction (rounded to two decimal places):

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General solution: $p_C = .94p_S$, $p_E = .85p_S$ and p_S is free.

Equilibrium price vector:

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Restriction: choose p_S non-negative, e.g.: $p_S =$ \$100 million, $p_C =$ \$94 million, $p_E =$ \$85 million.

Balancing Chemical Equations

Propane gas burning:

$$\mathrm{C_3H_8}\text{ (propane)} + \mathrm{O_2}\text{ (oxygen)} \rightarrow$$
$$\mathrm{CO_2}\text{ (carbon dioxide)} + \mathrm{H_2O}\text{ (water)}$$

Problem: how much propane and oxygen (exactly) produce how much carbon dioxide and water? Or:

$$(x_1)\mathrm{C_3H_8} + (x_2)\mathrm{O_2} \rightarrow (x_3)\mathrm{CO_2} + (x_4)\mathrm{H_2O}$$

Additional restriction: molecule amounts (x_i) must be integral

Solution: for each molecule (reactants and products), con-

struct "atom vector" of atoms per molecule:

$$\begin{bmatrix} \text{Carbon} \\ \text{Hydrogen} \\ \text{Oxygen} \end{bmatrix}$$

We have:

$$\mathrm{C_3H_8} : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \quad \mathrm{O_2} : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{CO_2} : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathrm{H_2O} : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Balancing, we require:

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Rearranging:

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduction . . . :

$$x_1 = \frac{1}{4}x_4, \quad x_2 = \frac{5}{4}x_4, \quad x_3 = \frac{3}{4}x_4, \quad x_4 \text{ free}$$

Due to integral constraint, $x_4 := 4$ and we have:

$$\mathrm{C_3H_8} + 5\mathrm{O_2} \to 3\mathrm{CO_2} + 4\mathrm{H_2O}$$

Note: since x_4 free, there exist an infinite number of solutions, i.e., every integral mutiple of the above.

Network Flow

Studying flow through network:

1) urban planners and traffic engineers: traffic flow through city streets
2) electrical engineers: current flow through circuits
3) economists: distribution of products from manufacturers to comsumers, through wholesalers and retailers

System can have 100s or 1000s of variables and equations

Network:

1) nodes (or junctions)
2) branches connecting some or all nodes

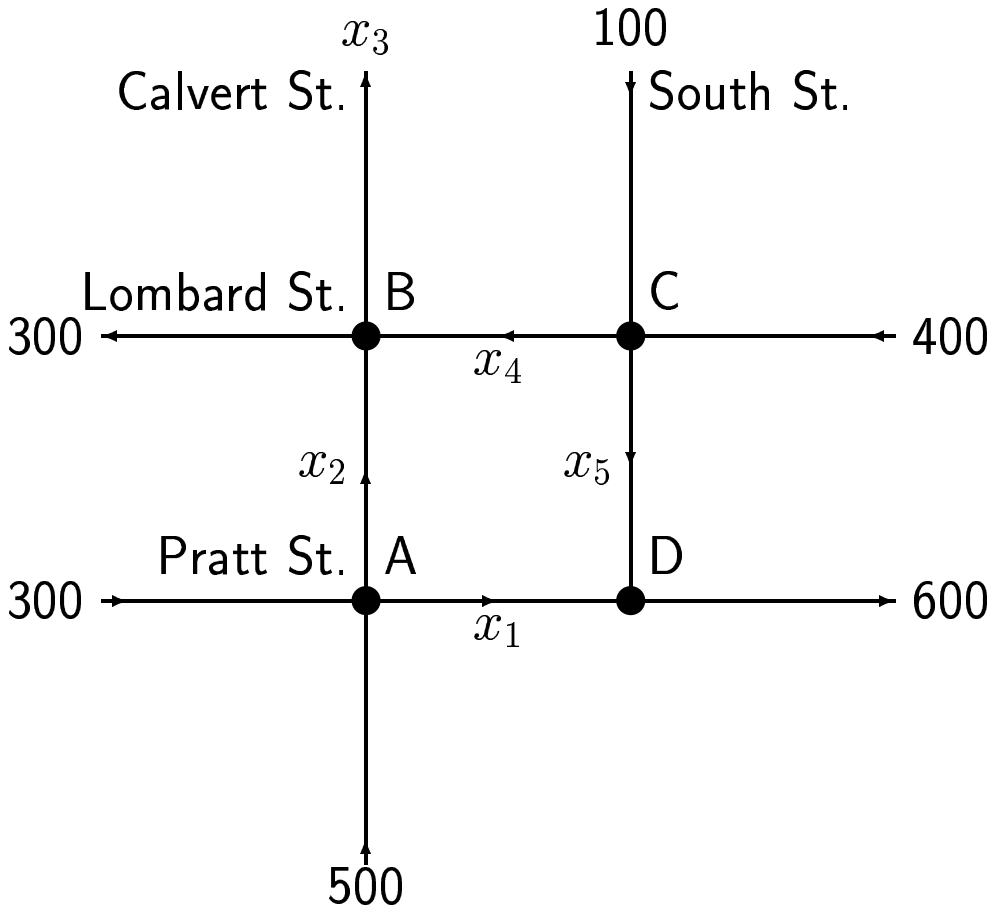
Flow:

1) amount or rate
2) along each branch
3) directed

4) known or variable (x_i)

Conservation assumption for network *and* each node:

$$\text{flow in} = \text{flow out}$$



Conservation assumptions at nodes:

Intersection	Flow in	=	Flow out
A	300 + 500	=	$x_1 + x_2$
B	$x_2 + x_4$	=	300 + x_3
C	100 + 400	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

Conservation assumption for entire network:

$$500 + 300 + 100 + 400 = 300 + x_3 + 600$$

or: $x_3 = 400$

We have:

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & & & = & 800 \\ & & x_2 & - & x_3 & + & x_4 & = & 300 \\ & & & & & & x_4 + x_5 & = & 500 \\ x_1 & & & & & & + x_5 & = & 600 \\ & & & & x_3 & & & = & 400 \end{array}$$

Row reduction of augmented matrix:

$$\begin{array}{rcrcrcrcl} x_1 & & & & & & + x_5 & = & 600 \\ & & x_2 & & & & - x_5 & = & 200 \\ & & & & x_3 & & & = & 400 \\ & & & & & & x_4 + x_5 & = & 500 \end{array}$$

General pattern flow:

$$\begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$

Note:

1) negative flow means in the opposite direction
2) limitation here: not posssible, since only one ways
3) leads to limitations, e.g.: $x_5 \leq 500$
4) other limitations?